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LETTER TO THE EDITOR

The Z-transform method of evaluating partial summations in closed form

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Abstract. A method is shown to convert a partial summation into a residue problem using z-transforms. This method is particularly useful for evaluating partial sums whose infinite sums diverge.

The evaluation of partial summations in closed form is hindered by the lack of extensive series tables. Wheelon [1, 2] and Macfarlane [3] both describe methods to convert infinite summations to integrals using Laplace and Fourier–Mellin transforms, respectively. The extensive integral tables available are the rationale behind these methods. Consider the transform integral,

$$f(s) = \int F(t) K(s, t)\, \mathrm{d}t \tag{1}$$

where $K(s, t)$ is the kernel of the integral transform [4]. The partial sum S_n to be evaluated is given by

$$S_N = \sum_{s=1}^{N} f(s). \tag{2}$$

The summand is assumed to be a continuous function of s (even though it is evaluated at integer values) and is assumed to have a transform defined by (1) so that

$$S_N = \sum_{s=1}^{N} \int f(t) K(s, t)\, \mathrm{d}t. \tag{3}$$

Interchanging the summation and the integral (valid for finite sums),

$$S_N = \int f(t) \bar{k}(t)\, \mathrm{d}t \tag{4}$$

with

$$\bar{K}(t) = \sum_{s=1}^{N} K(s, t). \tag{5}$$

If (5) is summable in closed form, then the original sum (2) can be evaluated as an integral (4).

The method in this letter is similarly based on the z-transform [5–7] used in discrete processes and defined by

$$x(z)=\sum_{n=0}^{\infty} x(n)z^{-n} \tag{6}$$

$$x(n)=\frac{1}{2\pi\mathrm{i}}\oint x(z)z^{n-1}\,\mathrm{d}z. \tag{7}$$

Let the summand in (2) take the form of the transform in (7) so that

$$S_N=\sum_{n=1}^{N} x(n)=\frac{1}{2\pi\mathrm{i}}\oint x(z)\sum_{n=1}^{N} z^{n-1}\,\mathrm{d}z. \tag{8}$$

The summation in (8) is in the form of a geometric series,

$$S_N=\frac{1}{2\pi\mathrm{i}}\oint \frac{x(z)(z^N-1)}{z-1}\,\mathrm{d}z. \tag{9}$$

From the residue theorem [8], the value of the closed integral of an analytic function $f(z)$ is given by

$$\oint f(z)\,\mathrm{d}z=2\pi\mathrm{i}\sum \text{residues}\,[f(z)] \tag{10}$$

where the residues are evaluated by the singularities of $f(z)$ contained within the closed integral. For an mth-order pole at the location z_0, the residue is evaluated as [8]

$$\operatorname{Res} f(z)|_{z=z_0}\equiv\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}[(z-z_0)^m f(Z)]. \tag{11}$$

The integral representing the partial summation S_N (9) can be transformed using (10) into

$$S_N=\sum \text{residues}\left(\frac{x(z)(z^N-1)}{z-1}\right). \tag{12}$$

The problem of evaluating the partial sum (8) has been transformed into one of evaluating the residues of (12), using (11). As an example, consider the series

$$S_N=\sum_{n=1}^{N} n^3. \tag{13}$$

The z-transform of the summand is given by [5]

$$x(z)=\frac{z(z^2+4z+1)}{(z-1)^4}. \tag{14}$$

Using this transform in (12) produces a 5th-order pole at $z=1$ so that the residue is given by (using (11))

$$S_N=\frac{1}{4!}\frac{\mathrm{d}^4}{\mathrm{d}z^4}(z^{3+N}+4z^{2+N}+z^{N+1}-z^3-4z^2-z)_{z=1}$$

$$S_N=\tfrac{1}{4}(N+1)^2N^2$$

in agreement with tabulated values [9]. The advantage of this approach over the transform methods discussed previously [1-3] is in evaluating partial summations whose infinite summations are divergent, as is the case with (13). Neither the Laplace nor the Fourier–Mellin transform of the summand in equation 13 exists. Another advantage of the z-transform method is that the residue form of the partial series is usually easier to evaluate than the integral form.

This letter has shown how the problem of evaluating a partial series can be reduced to a problem of solving residues. The procedure complements those based on the Laplace and Fourier–Mellin transforms, while broadening the number of series forms that may be evaluated analytically.

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